4.1 Exponential Functions

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April 9, 2011

Definitions

- The functions that involve some combinations of basic arithmetic operations, powers, or roots are called **algebraic functions**.
- Since the exponential and logarithmic functions transcend what can be described with algebraic functions, they are called **transcendental functions**.

Definition (Exponential Function)

An exponential function with base a is a function of the form

$$f(x) = a^x,$$

where a and x are real numbers and

- x is the independent VARIABLE of the function; and
- a is a number FIXED CONSTANT such that $a > 0$ and $a \neq 1$.

Examples:

$$f(x) = 2^x \qquad g(x) = 10^x$$

$$h(x) = \left(\frac{1}{2}\right)^x \qquad w(x) = \left(\frac{113}{10}\right)^x$$

The domain of $f(x) = 2^x$ is all real numbers. To understand how this function can be evaluated for any real number, first note that the function is evaluated for any rational number using powers and roots. For example,

$$f(1.7) = f(17/10) = 2^{17/10} = \sqrt[10]{2^{17}} \approx 3.249009585.$$

Consider the expression $2^{\sqrt{3}}$ and recall that the irrational number $\sqrt{3}$ is an infinite nonterminating nonrepeating decimal number:

$$\sqrt{3} = 1.7320508075...$$

If we use rational approximations to $\sqrt{3}$ as exponents, we see a pattern:

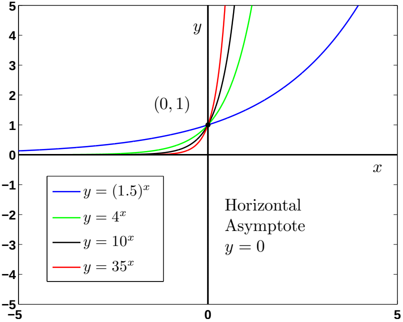
$$\begin{aligned} 2^{1.7} &= 3.249009585\ldots \\ 2^{1.73} &= 3.317278183\ldots \\ 2^{1.732} &= 3.321880096\ldots \\ 2^{1.73205} &= 3.321995226\ldots \\ 2^{1.7320508} &= 3.321997068\ldots \end{aligned}$$

As the exponents get closer and closer to $\sqrt{3}$ we get results that are approaching some number. We define $2^{\sqrt{3}}$ to be that number.

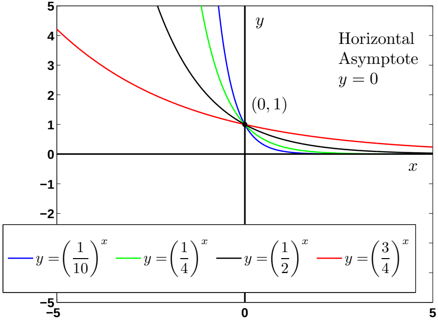
Of course it is impossible to write the exact value of $\sqrt{3}$ or $2^{\sqrt{3}}$ as a decimal, but you can use a calculator to get $2^{\sqrt{3}} \approx 3.321997085$. Since any irrational number can be approximated by rational numbers in this same manner, 2^x is defined similarly for any irrational number. This idea extends to any exponential function.

Theorem (domain of the exponential function)

The domain of $f(x) = a^x$ for $a > 0$ and $a \neq 1$ is the set of all real numbers.



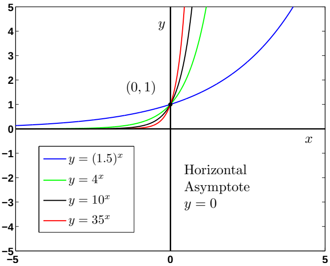
$a > 1$

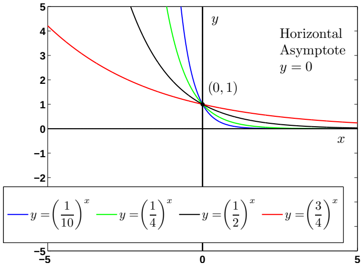


$0 < a < 1$

The exponential function $f(x) = a^x$ is only defined for $a > 1$ and $0 < a < 1$, and the graph of an exponential function can only exhibit two types of behavior:

1. exponential growth (if $a > 1$), or
2. exponential decay (if $0 < a < 1$).



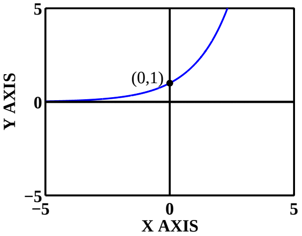


Properties of Exponential Functions

Every exponential function $f(x) = a^x$ has the following properties:

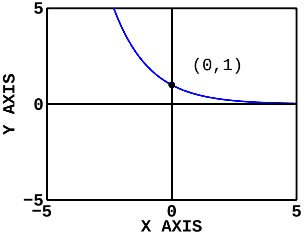
- *f(x)* is an increasing function whenever $a > 1$ and a decreasing function when $0 < a < 1$.
- The y-intercept of the graph of f is $(0, 1)$ for any a.
- The graph has the x-axis (y=0) as a horizontal asymptote (HA).
- The domain of f is $(-\infty, \infty)$, and the range of f is $(0, \infty)$.
- The function f is one-to-one, hence it passes the HORIZONTAL LINE TEST line test and is invertible.

$f(x) = 2^x$



$a > 1$

$f(x) = \left(\frac{1}{2}\right)^x$



$0 < a < 1$

$y = a^x$	$a > 1$	$0 < a < 1$
End Behavior	$\lim_{x\to\infty} 2^x = \infty$ $\lim_{x\to-\infty} 2^x = 0$	$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = 0$ $\lim_{x\to-\infty} \left(\frac{1}{2}\right)^x = \infty$
Horiz. Asymp. (HA)	$y = 0$	$y = 0$
domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
range:	$(0, \infty)$	$(0, \infty)$

Definition

Consider the function g defined by

$$g(x) = a \cdot f(x - c) + d \qquad \text{where } a, c, \text{ and } d \text{ are real numbers.}$$

Then

1. $g(x)$ is the "generalized" child graph of parent graph $f(x)$.
2. c represents the horizontal translation of f.
3. a represents the reflection/compression/expansion of f (Horizontally Speaking).
   a represents the reflection/stretching/shrinking of f (Vertically Speaking).
4. d represents the vertical translation of f.

Definition (Multiple Transformations Graphing Algorithm)

Consider the function g defined by

$$g(x) = a \cdot f(x - c) + d \quad \text{where } a, c, \text{ and } d \text{ are real numbers.}$$

In order to graph $g(x)$ it is recommended to take the following steps:

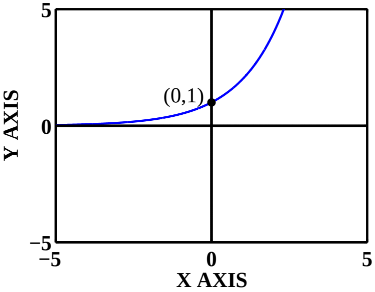
1. Identify and graph the parent graph $f(x)$, of $g(x)$.
2. (c) Translate (shift) f horizontally, i.e. apply $f(x \pm c)$.
3. (a) Reflect/compress/expand f (Horizontally Speaking).
   reflect/stretch/shrink f (Vertically Speaking).
4. (d) Translate (shift) f vertically.

Note: If you are asked to graph, for example, $f(x) = -2\sqrt[3]{x+1} - 2$, then you should rename $f(x)$ and give it the new name of $g(x)$. Then find g's parent graph $f(x)$.

Use translations to graph: $g(x) = -3 \cdot 2^{x-1} + 1$

Step 1: Identify and graph the parent function: $f(x) = 2^x$. We will select a few points on the parent graph to follow the evolution of, as we apply each step of the graphing algorithm.

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4



Step 2: Apply the horizontal shift. Graph:
$g_1(x) = 2^{x-1} = f(x-1)$ by shifting every point on the parent graph one unit right (horizontally).

Use translations to graph: $g(x) = -3 \cdot 2^{x-1} + 1$

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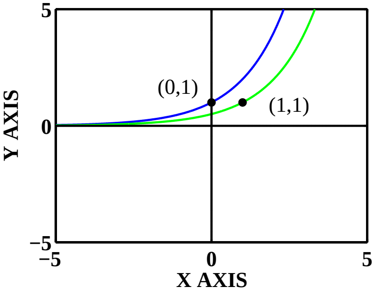
$g_1(x) = 2^{x-1} = f(x-1)$ by shifting every point on the parent graph one unit right (horizontally).

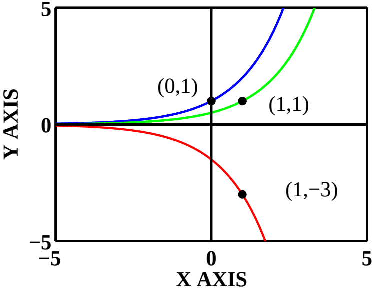
x	$g_1(x)$
-1	$\frac{1}{4}$
0	$\frac{1}{2}$
1	1
2	2
3	4



Step 3: Apply the reflection/magnification. Graph:
$g_2(x) = -3 \cdot 2^{x-1} = -3 \cdot f(x-1)$. Now shift the points on the graph in step 2 by multiplying the y coordinate of each point by -3.

Step 3: Apply the reflection/magnification. Graph: $g_2(x) = -3 \cdot 2^{x-1} = -3 \cdot f(x-1)$. Now shift the points on the graph in step 2 by multiplying the y coordinate of each point by -3.

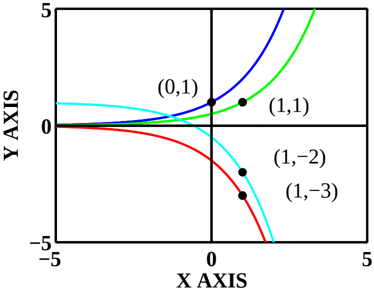
x	$g_2(x)$
-1	$-\frac{3}{4}$
0	$-\frac{3}{2}$
1	-3
2	-6
3	-12



Step 4: Apply the vertical shift now. Graph: $g(x) = -3 \cdot 2^{x-1} + 1 = -3 \cdot f(x-1) + 1$. Translate each point on the graph in step 3 vertically upwards. (NOTE: the horizontal asymptote (HA) shifts up one unit to $y = 1$)

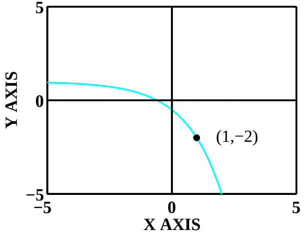
Step 4: Apply the vertical shift now. Graph: $g(x) = -3 \cdot 2^{x-1} + 1 = -3 \cdot f(x-1) + 1$. Translate each point on the graph in step 3 vertically upwards. (NOTE: the horizontal asymptote (HA) shifts up one unit to $y = 1$)

x	$g(x)$
-1	$\frac{1}{4}$
0	$-\frac{1}{2}$
1	-2
2	-5
3	-11



$g(x) = -3 \cdot 2^{x-1} + 1 = -3 \cdot f(x-1) + 1.$

$g(x) = -3 \cdot 2^{x-1} + 1 = -3 \cdot f(x-1) + 1$ has the following characteristics:



- $g(x)$ has a horizontal asymptote at $y = 1$
- dom(g)$\equiv (-\infty, \infty)$
- rng(g)$\equiv (-\infty, 1)$
- $g(x)$ is a decreasing function.
- $\lim_{x \to \infty} g(x) = -\infty$
- $\lim_{x \to -\infty} g(x) = 1$

Example: Find the exponential function $f(x) = C \cdot a^x$ whose graph goes through the point (0,4) and (3,32).

Soln: We need to determine what are the values of C and a. Substituting $(x, y) = (0, 4)$ into f renders

$$4 = C \cdot a^0 \implies C = 4 \text{ since } a^0 = 1$$

Now we know $f(x) = 4 \cdot a^x$. Substituting $(x, y) = (3, 32)$ into f renders

$$32 = 4 \cdot a^3 \implies 8 = a^3 \implies 2^3 = a^3$$

Taking cube roots of both sides of this last equation tells us that $a = 2$. Hence the desired function is $f(x) = 4 \cdot 2^x$

Theorem (Exponential Equality)

For $a > 0$ and $a \neq 1$,

$$\text{if } a^{x_1} = a^{x_2}, \text{ then } x_1 = x_2$$

also

$$\text{if } x_1 = x_2, \text{ then } a^{x_1} = a^{x_2}$$

Example: Solve $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ for x.

Soln: Try to use the above theorem.

$$\left(\frac{2}{3}\right)^x = \frac{9}{4} = \left(\frac{3}{2}\right)^2$$

or

$$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2} \text{ since } \left(\frac{3}{2}\right)^2 = \left(\frac{2}{3}\right)^{-2}$$

Then by the theorem on exponential equality, we must have that $x = -2$.